Maxwell’s Equations through the Major Vector Theorems

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0: Introduction

Maxwell’s Equations are four equations originally formulated by Carl Friedrich Gauss (1777-1855), Michael Faraday (1791-1867), André-Marie Ampère (1775-1836), and James Clerk Maxwell (1831–1879) that describe various properties of electric and magnetic fields. Here are all four of Maxwell’s Equations in their entirety.

Integral and differential forms of Gauss’s law for electric fields:

\[ \oint_S E \cdot \hat{n} \, dS = \frac{q_{\text{in}}}{\varepsilon_0} \]
\[ \nabla \cdot E = \frac{\rho}{\varepsilon_0} \]

Integral and differential forms of Gauss’s law for magnetic fields:

\[ \oint_S B \cdot \hat{n} \, dS = 0 \]
\[ \nabla \cdot B = 0 \]

Integral and differential forms of Faraday’s law:

\[ \oint_C E \cdot d\vec{r} = -\frac{d}{dt} \oint_S B \cdot \hat{n} \, dS \]
\[ \nabla \times E = -\frac{\partial B}{\partial t} \]

Integral and differential forms of the Ampere-Maxwell law:

\[ \oint_C B \cdot d\vec{r} = \mu_0 \left( I_{\text{enc}} + \varepsilon_0 \frac{d}{dt} \oint_S E \cdot \hat{n} \, dS \right) \]
\[ \nabla \times B = \mu_0 \left( \vec{J} + \varepsilon_0 \frac{\partial E}{\partial t} \right) \]

Right now these equations might look confusing and perhaps even meaningless. However, the purpose of this paper is to try to shed light on the various properties of these equations and their consequences. Sections 1-4 focus on providing the physical interpretation of each of Maxwell’s Equations in integral form. Sections 5-6 give a brief overview of Stokes’ Theorem and the Divergence Theorem from Calculus. Using these theorems, Sections 7-10 give a description of the processes used to derive the differential forms of Maxwell’s Equations from the integral forms. Sections 11-12 show how the wave equations of the electric field and magnetic field are derived by using the
differential forms of Maxwell’s Equations. Section 13 gives a description of the electromagnetic wave phenomenon and attempts to show how electric and magnetic waves truly unify into one. Our discussion of Maxwell’s Equations concludes in Section 14 wherein the speed of propagation of electromagnetic waves is derived, leading to a foundational result in the electromagnetic theory of light.

Within each section we will frequently refer to equations and expressions by simple names. Note that numerical names such as (1), (2) and (3) are reused from section to section. However, special names like (I-GEF), (ST2) and (WE-Gen) always represent a unique expression in the text, so any textual reference to an expression with this sort of name with letters and numbers might require the reader to recall the expression from a previous section.

### 1: Physical Meaning of Gauss’s Law for Electric Fields in Integral Form

First, we explain the physical meaning of the integral form of Gauss’s law for electric fields, which is given in (I-GEF) below.

**Integral form of Gauss’s law for electric fields:**

\[
\iint_S \mathbf{E} \cdot \hat{n} \, dS = \frac{q_{in}}{\varepsilon_0}
\]

First, let’s look at the left side of (I-GEF),

\[
(1) \quad \iint_S \mathbf{E} \cdot \hat{n} \, dS
\]

Here \( \mathbf{E} \) is an electric field and \( S \) is a closed surface. Expression (1) is defined as the total flux of the electric field \( \mathbf{E} \) across the surface \( S \).

Now we consider the right side of (I-GEF),

\[
(2) \quad \frac{q_{in}}{\varepsilon_0}
\]

This is a reasonably simple expression. \( q_{in} \) represents the total amount of electric charged enclosed in our surface \( S \). \( \varepsilon_0 \) is called the permittivity of free space and its value is \( 8.8541878176 \times 10^{-12} \, C/Vm \) (coulombs per volt-meter). This is a value derived from experiment, but we will not go into its physical meaning here.
Looking at the information from (1) and (2) given above, we now see that (I-GEF) states that the total flux of the electric field $\vec{E}$ across a closed surface $S$ is proportional to the total electric charge enclosed by the surface.

Hence given a closed surface, the more electric charge enclosed in the surface, the higher the electric flux passing through the surface. This leads to the result that if the closed surface contains no electric charge, then the total electric flux passing through that surface is zero. As a matter of fact, any electric field generated by charges lying outside of our closed surface will contribute nothing to the total electric flux through $S$, so we must only concern ourselves with charges enclosed in $S$, an idea evident by the $q_{\text{in}}$ term in (I-GEF). We see this concept in the figures below.

![Figure 1](image1.png) ![Figure 2](image2.png)

In both figures we have a charge that generates an electric field, represented by the arrows emanating from the charges. In Figure 1 we have a charge inside a closed surface, and we can definitely see that the charge inside the surface contributes to the total electric flux through the surface. But in Figure 2 we have a charge outside of a closed surface. Any electric field line that enters the surface must leave it as well, meaning that while we have a certain value for the electric flux at the point where the electric field line enters the surface, we will have the same value of electric flux at the point where the line leaves the surface, except for a reversal of sign. So, every component of the electric field generated by the charge contributes nothing to the total electric flux across the surface. Therefore, we see that only charges internal to the surface contribute to the flux of the electric field through that surface.
2: Physical Meaning of Gauss’s Law for Magnetic Fields in Integral Form

Next we explain the physical meaning of the integral form of Gauss’s law for magnetic fields, which is given in (I-GMF) below.

Integral form of Gauss’s law for magnetic fields:

\[
\oiint_S \mathbf{B} \cdot \hat{n} \, dS = 0
\]

Here \( \mathbf{B} \) is a magnetic field and \( S \) is a closed surface. (I-GMF) states that the flux of the magnetic field \( \mathbf{B} \) through a closed surface \( S \) is zero. To illustrate why this is so, we refer to Figure 3 below.

† Figure 3

Here we see some closed surfaces in a magnetic field. Notice that the magnetic field lines appear in loops. Unlike electric field lines which originate or terminate on charges (depending on whether the charge is positive or negative), magnetic field lines always loop back on themselves. This property means that any magnetic field line that enters a closed surface must leave it as well. Since any magnetic field like that enters the surface also leaves the surface at some point, the total magnetic flux across the closed surface is zero; one can refer to the explanation of Figure 2 to see why this is so. This verifies (I-GMF).

† A similar illustration is found in (Fleisch [3, p. 49])
3: Physical Meaning of Faraday’s Law in Integral Form

Now we explain the physical meaning of the integral form of Faraday’s law, shown as (I-F) below.

**Integral form of Faraday’s law:**

\[
\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot \mathbf{n} \, dS
\]

First, let us look at the left side of this expression, (1)

\[
\oint_C \mathbf{E} \cdot d\mathbf{r}
\]

Here \( \mathbf{E} \) is an electric field with \( C \) being a closed path. (1) is the circulation of the electric field \( \mathbf{E} \) along \( C \).

Next we look at the right side of (I-F),

\[
-\frac{d}{dt} \iint_S \mathbf{B} \cdot \mathbf{n} \, dS
\]

Here \( \mathbf{B} \) is a magnetic field and \( S \) is a surface. However, since \( S \) is involved with (I-F), it is worth noting that the surface must have the aforementioned closed path \( C \) as a boundary; other than this condition, the surface is arbitrary. Hence (2) is the negative of the derivative of the flux of the magnetic field \( \mathbf{B} \) across the surface \( S \) with respect to time.

Therefore, by combining (1) and (2) we can see that (I-F) states that the circulation of the electric field \( \mathbf{E} \) along a closed curve \( C \) is equal to the negative of the derivative of the flux of the magnetic field \( \mathbf{B} \) across the surface \( S \) with respect to time, where \( S \) is any surface bounded by \( C \).

To understand how Faraday’s law works, we refer to some interesting principles. First we explain how an electric current produces a magnetic field by referring to Figure 4 below.
We can see from Figure 4 that an electric current through a wire will induce a magnetic field that appears in loops around the wire. The direction of this induced magnetic field is determined by the right-hand rule. This property will be important later.

In order to see how the components of (I-F) work together, we refer to Figure 5 below.

Here we can see our surface $S$ bounded by a closed loop $C$. The magnetic field lines going through $S$ produce a magnetic flux through $S$. And, since our magnet is moving toward the surface, the magnetic flux through the surface is increasing with time. Hence (I-F) tells us that the circulation of the electric field $\vec{E}$ around $C$ will be nonzero; an electric field circulation is induced along $C$. But how do we tell what direction this circulation is in? In Figure 5 we see that the leftward magnetic flux through $S$ is increasing over time. Notice that in (I-F) the circulation of the electric field $\vec{E}$ around $C$ is equal to the negative value of the rate of change of the magnetic flux through $S$ over time. This means that our electric field circulation over $C$ should create a magnetic field that opposes the changing magnetic flux through $S$. Hence, the circulation of the electric field over $C$ should produce rightward magnetic flux through $S$, since the magnet is producing increasing leftward flux.
With Figure 4 we showed how a magnetic field is produced from a flowing electric current. We can apply this to an electric field circulation. Here we want the circulation to produce rightward magnetic flux. That is, if we think of the electric field circulation over $C$ as an electric current running through $C$, then that current should produce a magnetic field that forms in loops around $C$, and those loops should produce rightward magnetic flux through $S$ as they penetrate $S$. Using the right hand rule, we see that the circulation of $\vec{E}$ over $C$ is in the direction displayed in Figure 5. Hence we have described the physical interpretation of $(I-F)$.

† A similar illustration is found in (Fleisch [3. p.71])

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4: Physical Meaning of the Ampere-Maxwell Law in Integral Form

Here we explain the physical meaning of the integral form of the Ampere-Maxwell law, shown as $(I-AM)$ below.

Integral form of the Ampere-Maxwell law:

$$(I-AM) \quad \oint_C \vec{B} \cdot d\vec{r} = \mu_0 \left( I_{enc} + \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \hat{n} \, dS \right)$$

First we look at the left side of $(I-AM)$,

$$(1) \quad \oint_C \vec{B} \cdot d\vec{r}$$

Here $\vec{B}$ is a magnetic field and $C$ is a closed path. Thus $(1)$ is the circulation of the magnetic field $\vec{B}$ along the closed path $C$. On the right side of $(I-AM)$,

$$(2) \quad \mu_0 \left( I_{enc} + \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \hat{n} \, dS \right)$$

we notice many as of yet undefined terms that must be introduced.

We should recognize the term $\varepsilon_0$ from the integral form of Gauss’s law for electric fields as “the permittivity of free space”. Similar to $\varepsilon_0$, $\mu_0$ is another experimental physical constant called “the permeability of free space”. Its value is $\mu_0 = 4\pi \times 10^{-7} \text{Vs/Am}$ (Volt-seconds per Ampere-meter). Once again, we refrain from going into its physical meaning.
\( I_{\text{enc}} \) is the total electric current enclosed by \( C \). Since \( C \) is a closed loop, we can draw an arbitrary surface \( S \) that is bounded by \( C \). Thus we define \( I_{\text{enc}} \) as the total electric current that runs through this arbitrary surface.

In (2) it is also useful to note the expression

\[
(3) \quad \frac{d}{dt} \int_S \mathbf{E} \cdot \hat{n} \, dS
\]

Here \( \mathbf{E} \) is an electric field and \( S \) is a surface. Since \( S \) here is picked with regard to (I-AM), \( S \) must be bounded by the aforementioned closed path \( C \); other than this requirement, \( S \) is arbitrary. Thus (3) is the derivative of the flux of the electric field \( \mathbf{E} \) across the surface \( S \) with respect to time, where \( S \) is any surface bounded by the aforementioned closed path \( C \).

Now that we have defined the different terms of (2), it is apparent that (I-AM) states that the circulation of the magnetic field \( \mathbf{B} \) along a closed path \( C \) is proportional to the total electric current enclosed by \( C \) plus \( \varepsilon_0 \) times the derivative of the flux of the electric field \( \mathbf{E} \) across the surface \( S \) with respect to time, where \( S \) is an arbitrary surface that is bounded by \( C \). Physically, this means that for any surface \( S \) with closed boundary path \( C \), either an enclosed electric current by \( C \) or a changing electric flux across \( S \) will induce a magnetic field circulation along \( C \).

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**5: Stokes’ Theorem**

† Let \( S \) be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve \( C \) with positive orientation. Let \( \mathbf{F} \) be a vector field whose components have continuous partial derivatives on an open region in \( \mathbb{R}^3 \) that contains \( S \). Then

\[
(\text{ST1}) \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot dS
\]

Since \( dS = \hat{n} dS \), we can also write the theorem this way:

\[
(\text{ST2}) \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot \hat{n} dS
\]

First, let us investigate the left side of (ST2),

\[
(1) \quad \oint_C \mathbf{F} \cdot d\mathbf{r}
\]
Here $\vec{F}$ is a vector field and $C$ is a closed path. Hence (1) is defined as the circulation of $\vec{F}$ along $C$.

Now, let us investigate the right side of (ST2),

\[
(2) \quad \iint_S \text{curl} \vec{F} \cdot \hat{n} \, dS
\]

Here we have the same vector field $\vec{F}$ as before, but now we refer to the surface $S$ as well. The surface $S$ is completely arbitrary, as long as it has the aforementioned closed path $C$ as a boundary. Also, $\hat{n}$ is a unit normal vector with respect to $S$. Thus (2) is the flux of the curl of $\vec{F}$ across $S$.

Combining what has been said about (1) and (2), Stokes’ Theorem says that the circulation of $\vec{F}$ over $C$ is equal to the flux of the curl of $\vec{F}$ across $S$. The vector field $\vec{F}$, the path $C$, and the surface $S$ as all defined as they were in our initial statement of Stokes’ Theorem at the beginning of this section.

It is easy to see why Stokes’ Theorem holds true by referring to a visual reference which is shown in Figure 6 below.

Here we see a cutout of the surface $S$ that lies along the boundary $C$ of $S$, where $C$ is a closed path. Each of the small squares surrounds a point on $S$, and each square contains a circulating set of arrows, representing the flux of the curl of $\vec{F}$ at a point on $S$. So, if we let these squares be infinitesimally small, cover all of $S$, and add all of these “flux of curls” together, we would essentially get the total flux of the curl of $\vec{F}$ across $S$, which is precisely the right side of Stokes’ Theorem, (2).
We can see that the flux of the curl of $\mathbf{F}$ at each point cancels out the flux of the curl of $\mathbf{F}$ at each adjacent point since the arrows will run in opposite directions between adjacent points. The only places where the arrows will not cancel out are along the boundary curve $C$ of $S$. Where the arrows do not cancel out, we obtain the circulation of $\mathbf{F}$ along $C$, represented by the red arrows in the diagram; this is precisely the left side of Stokes' Theorem, (1).

Combining what we have said, it is clear that the circulation of $\mathbf{F}$ along $C$ is equal to the total flux of the curl of $\mathbf{F}$ across $S$. Thus we can see how Stokes’ Theorem can be represented in a diagram.

† Definition taken from (Stewart [4, p1157])
‡ A similar illustration is found in (Fleisch [3, p. 116])

6: The Divergence Theorem

† Let $E$ be a simple solid region and let $S$ be the boundary surface of $E$, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains $E$. Then

\[
\int\int\int\int E \text{div} \mathbf{F} \, dV = \iiint_S \mathbf{F} \cdot d\mathbf{S} \tag{DT1}
\]

Since $d\mathbf{S} = \mathbf{n} \, dS$, we can also write the theorem this way:

\[
\int\int\int\int E \text{div} \mathbf{F} \, dV = \iiint_S \mathbf{F} \cdot \mathbf{n} \, dS \tag{DT2}
\]

First, we investigate the left side of (DT2),

\[
\iiint_S \mathbf{F} \cdot \mathbf{n} \, dS \tag{1}
\]

Here $\mathbf{F}$ is a vector field, $S$ is a closed surface, and $\mathbf{n}$ is a unit normal vector with respect to $S$. Thus (1) is the flux of $\mathbf{F}$ across $S$.

Now we investigate the right side of (DT2),
Here \( \mathbf{F} \) is the same vector field as before, and \( E \) is the volume bounded by the aforementioned closed surface \( S \). Thus (2) is the volume integral over \( E \) of the divergence of \( \mathbf{F} \).

Combining (1) and (2) it is now clear that the Divergence Theorem states that the flux of the vector field \( \mathbf{F} \) across \( S \) is equal to the volume integral over \( E \) of the divergence of \( \mathbf{F} \). Here \( \mathbf{F} \), \( S \), and \( E \) are defined in the same way as our initial definition of the Divergence Theorem at the beginning of this section.

To see why the Divergence Theorem holds true, it is helpful to look at a visual representation which is given below in Figure 7.

Each of the small cubes inside the volume surrounds a point in the solid \( E \), and the arrows emanating from each cube represents the divergence of the vector field \( \mathbf{F} \) at that point in \( E \). If we let these cubes be infinitesimally small, covering all of \( E \), then we can see that all of these divergences added together should equal the volume integral over \( E \) of the divergence of \( \mathbf{F} \). This is the right side of the Divergence Theorem, (2).

Notice that the arrows, or divergences, run in opposite directions for adjacent points inside the volume. Only along the boundary surface \( S \) of the volume are the arrows not canceled out by adjacent points. The sum of all the arrows that pass through \( S \) constitutes the flux of the vector field \( \mathbf{F} \) across \( S \), which is the left side of the Divergence Theorem, (1).

Combining what has been said so far, it should be clear that the flux of the vector field \( \mathbf{F} \) across \( S \) is equal to the volume integral over \( E \) of the divergence of \( \mathbf{F} \). Hence we can see how the Divergence Theorem can be represented in a diagram.
7: Derivation of the Differential Form of Gauss’s Law for Electric Fields

Here we will show how to derive the differential form of Gauss’s law for electric fields from the integral form. That is, we show that, through the Divergence Theorem, (I-GEF) implies (D-GEF), with (I-GEF) and (D-GEF) given below.

Integral form of Gauss’s law for electric fields:

(I-GEF) \[ \iint_S \vec{E} \cdot \hat{n} \, dS = \frac{q_{in}}{\varepsilon_0} \]

Differential form of Gauss’s law for electric fields:

(D-GEF) \[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \]

First, we consider the left hand side of (I-GEF),

(1) \[ \iint_S \vec{E} \cdot \hat{n} \, dS \]

(1) is in the form needed to apply the Divergence Theorem. Thus, we see that

(2) \[ \iint_S \vec{E} \cdot \hat{n} \, dS = \iiint_E \text{div} \vec{E} \, dV \]

And since \( \text{div} \vec{E} = \nabla \cdot \vec{E} \), we can write

(3) \[ \iint_S \vec{E} \cdot \hat{n} \, dS = \iiint_E \nabla \cdot \vec{E} \, dV \]

Now, consider the right hand side of (I-GEF),

(4) \[ \frac{q_{in}}{\varepsilon_0} \]

It is defined that
Here $q_{in}$ is defined as the total electric charge (in coulombs) enclosed in the volume $E$ bounded by our closed surface $S$. $\rho$ is defined as the electric charge density (in coulombs per cubic meter) at various points throughout the volume $E$ (Note that $\rho$ can take on different values at different points in $E$). Hence (5) states that for our enclosed volume $E$, the total enclosed electric charge is equal to the volume integral of the electric charge density over $E$. Divide both sides of (5) by $\varepsilon_0$ and we get

$$\frac{q_{in}}{\varepsilon_0} = \iiint_E \frac{\rho}{\varepsilon_0} dV$$

Combining (I-GEF), (3) and (6) we now have

$$\iiint_E \nabla \cdot \vec{E} \, dV = \iiint_E \frac{\rho}{\varepsilon_0} \, dV$$

Since the solid $E$ was arbitrary, we know that the values inside the volume integrals are equal. Hence we arrive at

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

Thus, we can derive the differential form of Gauss’s law for electric fields from its integral form by applying the Divergence Theorem. What (D-GEF) says is that the divergence of the electric field at any point in space is equal to the electric charge density at that point divided by the permittivity of free space.

8: Derivation of the Differential Form of Gauss’s Law for Magnetic Fields

Here we show how to derive the differential form of Gauss’s law for magnetic fields from its integral from. That is, we show that (I-GMF) implies (D-GMF) by application of the Divergence Theorem, with equations (I-GMF) and (D-GMF) shown below.

Integral form of Gauss’s law for magnetic fields:

$$\iiint_S \vec{B} \cdot \hat{n} \, dS = 0$$

Differential form of Gauss’s law for magnetic fields:
This one is easy. Looking at the left side of (I-GMF),

\[ \oint_S \mathbf{B} \cdot \hat{n} \, dS \]

we can apply the Divergence Theorem to show that

\[ \iiint_E \operatorname{div} \mathbf{E} \, dV = \int_S \mathbf{B} \cdot \hat{n} \, dS \]

And since \( \operatorname{div} \mathbf{E} = \nabla \cdot \mathbf{E} \), we can write

\[ \iiint_E \nabla \cdot \mathbf{B} \, dV = \int_S \mathbf{B} \cdot \hat{n} \, dS \]

Now consider the right hand side of (I-GMF). We note that a triple integral of zero is still zero. That is,

\[ 0 = \iiint_E 0 \, dV \]

Combining (I-GMF), (3) and (4) we know that

\[ \iiint_E \nabla \cdot \mathbf{B} \, dV = \iiint_E 0 \, dV \]

Since the solid \( E \) was arbitrary, we know that the values inside the volume integrals are equal. Hence we arrive at

\[ \nabla \cdot \mathbf{B} = 0 \]

Thus we have shown that by application of the Divergence Theorem, the differential form of Gauss’s law for magnetic fields can be derived from its integral form. What (D-GMF) says is that the divergence of the magnetic field at any point in space is equal to zero.

9: Derivation of the Differential Form of Faraday’s Law

Here we show how to derive the differential form of Faraday’s law from its integral form. That is, by application of Stokes’ Theorem, we show that (I-F) implies (D-F), with (I-F) and (D-F) shown below.

Integral form of Faraday’s law:
Differential form of Faraday’s law:

\[(D-F)\quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\]

First we consider the left hand side of \((I-F)\),

\[(1)\quad \oint_{C} \vec{E} \cdot d\vec{r}\]

Applying Stokes’ Theorem we can see that

\[(2)\quad \oint_{C} \vec{E} \cdot d\vec{r} = \iint_{S} \text{curl} \ \vec{E} \cdot \hat{n} dS\]

And since \(\text{curl} \ \vec{E} = \nabla \times \vec{E}\) we can write \((2)\) as

\[(3)\quad \oint_{C} \vec{E} \cdot d\vec{r} = \iint_{S} (\nabla \times \vec{E}) \cdot \hat{n} dS\]

Combining \((I-F)\) with \((3)\), we arrive at the following equation:

\[(4)\quad \iint_{S} (\nabla \times \vec{E}) \cdot \hat{n} dS = -\frac{d}{dt} \iint_{S} \vec{B} \cdot \hat{n} dS\]

We pull the differential operator inside the integral on the right hand side of \((4)\) to obtain

\[(5)\quad \iint_{S} (\nabla \times \vec{E}) \cdot \hat{n} dS = \iint_{S} \left(-\frac{\partial \vec{B}}{\partial t}\right) \cdot \hat{n} dS\]

Since the surface \(S\) was arbitrary, we know that the values inside the surface integrals are equal. Hence we end up with

\[(D-F)\quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\]

Thus we have shown that by application of Stokes’ Theorem, \((I-F)\) implies \((D-F)\). That is, one can use Stokes’ Theorem to derive the differential form of Faraday’s law from its integral form. What \((D-F)\) says is that the curl of the electric field at any point in space is
equal to the negative value of the derivative of the magnetic field with respect to time at that point.

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10: Derivation of the Differential Form of the Ampere-Maxwell Law

Here we show how to derive the differential form of the Ampere-Maxwell law from its integral form. That is, by application of Stokes’ Theorem we show that \( \text{(I-AM)} \) implies \( \text{(D-AM)} \), with \( \text{(I-AM)} \) and \( \text{(D-AM)} \) shown below.

**Integral form of the Ampere-Maxwell law:**

\[
\oint_C \vec{B} \cdot d\vec{r} = \mu_0 \left( I_{enc} + \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \hat{n} \, dS \right)
\]

**Differential form of the Ampere-Maxwell law:**

\[
\nabla \times \vec{B} = \mu_0 \left( \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)
\]

First, we consider the left hand side of \( \text{(I-AM)} \),

\[
\oint_C \vec{B} \cdot d\vec{r}
\]

Using Stokes’ Theorem we see that

\[
\oint_C \vec{B} \cdot d\vec{r} = \int_S \text{curl} \, \vec{B} \cdot \hat{n} \, dS
\]

Since \( \text{curl} \, \vec{B} = \nabla \times \vec{B} \) we can write \( \text{(2)} \) as

\[
\oint_C \vec{B} \cdot d\vec{r} = \int_S \left( \nabla \times \vec{B} \right) \cdot \hat{n} \, dS
\]

Now consider the right hand side of \( \text{(I-AM)} \),

\[
\mu_0 \left( I_{enc} + \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \hat{n} \, dS \right)
\]

First we distribute things to make the expression \( \text{(4)} \) more understandable:
Next, it is defined that

\[ I_{\text{enc}} = \oint_S \mathbf{J} \cdot \mathbf{n} \, dS \]

Here, \( I_{\text{enc}} \) is the total electric current (in amperes) flowing through a surface \( S \) bounded by a closed path \( C \) (our surface \( S \) can be any surface we like, as long as its boundary is \( C \)). \( \mathbf{J} \) is the electric current density (in amperes per square meter) at various points on our surface \( S \) (Note that \( \mathbf{J} \) can take on different values at different points on \( S \)).

At any given point on \( S \), \( \mathbf{J} \) may or may not be flowing perpendicular to \( S \). This is because \( \mathbf{J} \) measures the vector current flowing through some arbitrarily small area perpendicular to our current \( I \), and \( I \) doesn’t necessarily need to run perpendicular to our surface \( S \). (We find \( \mathbf{J} \) at a point by letting our area surrounding the point, perpendicular to \( I \), be infinitesimally small.) To fix this problem, we take the dot product of \( \mathbf{J} \) and \( \mathbf{n} \), where \( \mathbf{n} \) is a unit normal vector to \( S \). Hence \( \mathbf{J} \cdot \mathbf{n} \) is the normal component of \( \mathbf{J} \) with respect to \( S \). Thus we can see that the surface integral of the normal components of the electric current density over our surface \( S \) is equal to the total electric current flowing through the surface \( S \). Hence (6) is true. Equation (6) states that the total current enclosed by a closed loop \( C \) is equal to the flux of the electric current density across an arbitrary surface \( S \) bounded by the loop \( C \).

Moving on, we can insert (6) into (5) to obtain

\[ \mu_0 \left( I_{\text{enc}} + \varepsilon_0 \frac{d}{dt} \iint_S \mathbf{E} \cdot \mathbf{n} \, dS \right) = \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d}{dt} \iint_S \mathbf{E} \cdot \mathbf{n} \, dS \]

In one step, we take the right side of (7) and pull everything inside one integral:

\[ \mu_0 \left( I_{\text{enc}} + \varepsilon_0 \frac{d}{dt} \iint_S \mathbf{E} \cdot \mathbf{n} \, dS \right) = \iint_S \left( \mu_0 \mathbf{J} \cdot \mathbf{n} \right) + \mu_0 \varepsilon_0 \left( \frac{\partial \mathbf{E}}{\partial t} \right) \cdot \mathbf{n} \, dS \]

By realizing that we want the normal components of \( \mathbf{J} \) and \( \mathbf{E} \), we can write (8) as

\[ \mu_0 \left( I_{\text{enc}} + \varepsilon_0 \frac{d}{dt} \iint_S \mathbf{E} \cdot \mathbf{n} \, dS \right) = \iint_S \left( \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot \mathbf{n} \, dS \]
Finally, we factor out $\mu_0$ on the right side of (9):

$$
(10) \quad \mu_0 \left( I_{enc} + \epsilon_0 \frac{d}{dt} \iint_S \mathbf{E} \cdot \hat{n} \, dS \right) = \iint_S \mu_0 \left( \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot \hat{n} \, dS
$$

We combine (I-AM), (3) and (10) to obtain

$$
(11d) \quad \iint_S (\nabla \times \mathbf{B}) \cdot \hat{n} \, dS = \iint_S \mu_0 \left( \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot \hat{n} \, dS
$$

Since the surface $S$ was arbitrary, we know that the values inside the surface integrals are equal. Hence we end up with

$$
(D-AM) \quad \nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)
$$

Thus we have shown that by application of Stokes’ Theorem, (I-AM) implies (D-AM). That is, Stokes’ Theorem shows that we can derive the differential form of the Ampere-Maxwell law from its integral form. What (D-AM) says is that the curl of the magnetic field at any point in space is equal to the electric current density plus the permittivity of free space multiplied by the derivative of the electric field with respect to time, all multiplied by the permeability of free space, at that point.

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11: Derivation of the Wave Equation of the Electric Field

Our derivation will take place in several steps. First, we take the curl of both sides of the differential form of Faraday’s law (D-F).

$$
(1) \quad \nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left( - \frac{\partial \mathbf{B}}{\partial t} \right)
$$

We pull the curl operator $\nabla \times$ inside of the differential $\frac{\partial \mathbf{B}}{\partial t}$. We can do this because $\mathbf{B}$ is assumed to be a sufficiently smooth vector field.

$$
(2) \quad \nabla \times (\nabla \times \mathbf{E}) = - \frac{\partial (\nabla \times \mathbf{B})}{\partial t}
$$

Now we shall make use of the vector field identity,
\( \nabla \times (\nabla \times E) = \nabla (\nabla \cdot E) - \nabla^2 E \)

We combine (2) and (3) to get

\( \nabla (\nabla \cdot E) - \nabla^2 E = -\frac{\partial (\nabla \times B)}{\partial t} \)

Using the differential form of the Ampere-Maxwell law (D-AM), we can substitute for the curl of the magnetic field \( \nabla \times B \).

\( \nabla (\nabla \cdot E) - \nabla^2 E = -\frac{\partial \left[ \mu_0 \left( \mathbf{J} + \epsilon_0 \left( \frac{\partial \mathbf{E}}{\partial t} \right) \right) \right]}{\partial t} \)

Using the differential form of Gauss’s law for electric fields (D-GEF), we can substitute for the divergence of the electric field \( \nabla \cdot \mathbf{E} \).

\( \nabla \left( \frac{\rho}{\epsilon_0} \right) - \nabla^2 E = -\frac{\partial \left[ \mu_0 \left( \mathbf{J} + \epsilon_0 \left( \frac{\partial \mathbf{E}}{\partial t} \right) \right) \right]}{\partial t} \)

Now we simplify by splitting up the differential on the right side of the equation.

\( \nabla \left( \frac{\rho}{\epsilon_0} \right) - \nabla^2 E = -\mu_0 \frac{\partial \mathbf{J}}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \)

We move all the terms pertaining to the electric field \( \mathbf{E} \) to the left side of the equation while moving all the other terms to the right side. Then we multiply both sides of the resulting equation by -1.

\( \nabla^2 E - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial \mathbf{J}}{\partial t} + \nabla \left( \frac{\rho}{\epsilon_0} \right) \)

We will now assume that we are working with a charge free, current free region. Since \( \rho \) pertains to charge density, this means that \( \rho = 0 \) in our case. Also, since \( \mathbf{J} \) is the electric current density, we have that \( \mathbf{J} = 0 \) as well. We make these substitutions into (8).

\( \nabla^2 E - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial (0)}{\partial t} + \nabla \left( \frac{0}{\epsilon_0} \right) = \mu_0 (0) + \nabla (0) = 0 + 0 = 0 \)

With simple algebra we end up with our conclusion.

The wave equation of the electric field:
12: Derivation of the Wave Equation of the Magnetic Field

Now we want to use a similar process to obtain the wave equation of the magnetic field. First we take the curl of both sides of the differential form of the Ampere-Maxwell law (D-AM).

\[(1) \quad \nabla \times (\nabla \times B) = \nabla \times \mu_0 \left( J + \varepsilon_0 \frac{\partial E}{\partial t} \right) \]

We once again take note of the vector field identity,

\[(2) \quad \nabla \times (\nabla \times B) = \nabla (\nabla \cdot B) - \nabla^2 B \]

Combining (1) and (2), we end up with (3) below.

\[(3) \quad \nabla (\nabla \cdot B) - \nabla^2 B = \nabla \times \mu_0 \left( J + \varepsilon_0 \frac{\partial E}{\partial t} \right) \]

We use the differential form of Gauss’s law for magnetic fields (D-GMF) to substitute for \( \nabla \cdot B \), then reduce.

\[(4) \quad \nabla (\nabla \cdot B) - \nabla^2 B = \nabla (0) - \nabla^2 B = -\nabla^2 B = \nabla \times \mu_0 \left( J + \varepsilon_0 \frac{\partial E}{\partial t} \right) \]

Once again we are going to assume that we are dealing with a charge free, current free region. Thus \( J = 0 \) and (4) becomes

\[(5) \quad -\nabla^2 B = \nabla \times \mu_0 \left( \varepsilon_0 \frac{\partial E}{\partial t} \right) \]

As before, we assume that our vector field \( E \) is sufficiently smooth so that we can move the curl operator \( (\nabla \times) \) inside the differential \( \frac{\partial E}{\partial t} \).

\[(6) \quad -\nabla^2 B = \mu_0 \left( \varepsilon_0 \frac{\partial (\nabla \times E)}{\partial t} \right) \]
Using the differential form of Faraday’s law (D-F), we substitute for the curl of $\mathbf{E}$ ($\nabla \times \mathbf{E}$).

\[
-\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial (-\frac{\partial \mathbf{B}}{\partial t})}{\partial t}
\]

(7)

We then simplify by rewriting the differentials.

\[
-\nabla^2 \mathbf{B} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}
\]

(8)

We reach our conclusion by multiplying both sides of (8) by -1.

The wave equation of the magnetic field:

\[
\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}
\]

(WE-MF)

13: A Description of Electromagnetic Waves

So far, we have come up with two separate wave equations, one for the electric field and one for the magnetic field. The former describes an oscillating electric field in the vacuum and the latter describes an oscillating magnetic field in the vacuum. However, it might seem odd that there is no such thing as an isolated electric wave in nature. The same applies to magnetic waves. Why is this? Consider an oscillating electric field. This electric field is changing with time. Not only this, but the vector field lines will change direction in a periodic manner. We refer to the integral form of the Ampere-Maxwell law below.

\[
\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 \left( I_{\text{enc}} + \varepsilon_0 \frac{d}{dt} \int_S \mathbf{E} \cdot \hat{n} \, dS \right)
\]

(I-AM)

Pick $S$ to be any surface with $C$ as its closed boundary such that $S$ does not vary with time. That is, the shape of our surface never changes as time passes. Since this equation holds for any surface $S$ with closed boundary, we can choose to consider only the case we have specified. We also assume that the enclosed current $I_{\text{enc}}$ by $C$ is zero, just as we did when deriving the wave equations for the electric field and magnetic field.

Since our electric field $\mathbf{E}$ is changing over time, (I-AM) tells us that it is going to induce a magnetic field circulation in $C$. If the flux of $\mathbf{E}$ across $S$ is increasing over time, the circulation of $\mathbf{B}$ over $C$ is going to be positive. Likewise, if the flux of $\mathbf{E}$ across $S$ is
decreasing over time, the circulation of $\mathbf{B}$ over $C$ is going to be negative. Since we defined $\mathbf{E}$ to be an oscillating electric field, the flux of $\mathbf{E}$ across $S$ is going to periodically switch between increasing and decreasing over time. Thus the circulation of $\mathbf{B}$ over $C$ is going to switch between being positive and negative as well. This is only possible if our magnetic field $\mathbf{B}$ is changing over time in an oscillatory manner. Therefore, an oscillating electric field gives rise to an oscillating magnetic field.

We could also prove that an oscillating magnetic field gives rise to an oscillating electric field by referring to the integral form of Faraday’s law instead,

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot \hat{n} \, dS$$

and using the same type of logical process as above. Thus an oscillating electric field gives rise to an oscillating field and vice versa. The fields always oscillate together in perfect phase.

Since electric and magnetic fields are completely entangled, there truly is no such thing as an electric wave or a magnetic wave by itself. The two fields oscillate in harmony and produce a sort of combination wave called an electromagnetic wave. The wave equation for the electric field (WE-EF) and the wave equation for the magnetic field (WE-MF) both describe electromagnetic waves since electromagnetic waves involve both an oscillating electric field and an oscillating magnetic field. As a matter of fact, we can completely describe an electromagnetic wave with either equation we choose, since one equation completely describes the other; this is because the electric and magnetic fields oscillate perfectly in phase. Thus, if we solve (WE-EF) and (WE-MF) for zero we come to this important result.

An electromagnetic wave is the result of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ oscillating together, and the electromagnetic wave equation can be written as either of the two second order, homogeneous partial differential equations listed below:

† The wave equation of the electromagnetic field:

$$(\text{WE-EMF:1}) \quad \left( \nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right) \mathbf{E} = 0$$

$$(\text{WE-EMF:2}) \quad \left( \nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right) \mathbf{B} = 0$$

The equations are second order because the order of the highest derivative is two, and they are homogeneous because we have set them to zero.
14: Deriving the Speed of Propagation of Electromagnetic Waves

We conclude our discussion by deriving a surprising consequence of the work we have done so far. First, we write down the general form of the wave equation for any vector field $\vec{A}$.

\[(\text{WE-Gen}) \quad \nabla^2 \vec{A} = \frac{1}{v^2} \frac{\partial^2 \vec{A}}{\partial t^2}\]

Here $v$ is the speed of propagation of the wave. We can compare (WE-Gen) to the wave equation of the electric field (WE-EF) or the wave equation of the magnetic field (WE-MF); remember that either equation completely describes the electromagnetic wave. We will go ahead and use (WE-EF), which is obviously equivalent to the wave equation of electromagnetic waves (WE-EMF:1).

*The wave equation of the electric field:*

\[(\text{WE-EF}) \quad \nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}\]

Comparing the general form of the wave equation (WE-Gen) to the wave equation of electric fields (WE-EF) we easily see that

\[(1) \quad \mu_0 \varepsilon_0 = \frac{1}{v^2}\]

Doing some algebra and substituting for the known values of $\mu_0 = 4\pi \times 10^{-7} \text{ m} \cdot \text{kg} / \text{C}^2$ and $\varepsilon_0 = 8.8541878 \times 10^{-12} \text{ C}^2 \cdot \text{s}^2 / \text{kg} \cdot \text{m}^3$, we end up solving for the speed of propagation for electromagnetic waves through the steps shown below. They should be self explanatory.

\[(2) \quad v^2 = \frac{1}{\mu_0 \varepsilon_0}\]

\[(3) \quad v = \sqrt[\mu_0 \varepsilon_0}\]

\[(4) \quad v = \sqrt{\frac{1}{(4\pi \times 10^{-7} \text{ m} \cdot \text{kg} / \text{C}^2)(8.8541878 \times 10^{-12} \text{ C}^2 \cdot \text{s}^2 / \text{kg} \cdot \text{m}^3)}}\]
\( v = \sqrt{8.987552 \times 10^{16} \text{ m}^2/\text{s}^2} = 2.9979 \times 10^8 \text{ m/s} = 299,792 \text{ km/s} \)

Therefore we see from (5) that the speed of propagation of an electromagnetic wave is 299,792 km/s (kilometers per second).

The value 299,792 km/s should be recognizable as the commonly known speed of light, an electromagnetic wave. This value agrees with the measured speed of light through experiment. Isn’t that amazing? After all of this work, Maxwell wrote that “light is an electromagnetic disturbance propagated through the field according to electromagnetic laws.” What makes Maxwell’s Equations so useful, then, is that they give us the wave equation for electromagnetic waves, and these waves are all around us in the form of radio waves, microwaves, gamma rays, x-rays, light, ultraviolet rays, infrared rays, and so on. The wave equation for electromagnetic waves can describe all of these, and we also know the speed at which all of these waves propagate. In other words, with the wave equation for electromagnetic waves we know the wave equation for all of these types of waves for free. Maxwell’s Equations, then, give us a basis for the electromagnetic theory of light and any other type of electromagnetic wave we want.

References


