ON THE RANGES OF DISCRETE EXPONENTIALS

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Received 3 December 2003

Let $a > 1$ be a fixed integer. We prove that there is no first-order formula $\phi(X)$ in one free variable $X$, written in the language of rings, such that for any prime $p$ with $\gcd(a, p) = 1$ the set of all elements in the finite prime field $F_p$ satisfying $\phi$ coincides with the range of the discrete exponential function $t \rightarrow a^t \pmod{p}$.

2000 Mathematics Subject Classification: 11T30, 11U09.

1. Introduction. Let $\phi(X)$ be a formula in one free variable $X$, written in the first-order language of rings. Then for every ring $R$ with identity, $\phi(X)$ defines a subset of $R$ consisting of all elements of $R$ satisfying $\phi(X)$. For example, the formula $(\exists Y)(X = Y^2)$ will define in every ring $R$ the set of perfect squares in $R$ (for an introduction to the basic concepts arising in model theory of first-order languages, we refer to [5]).

The value sets (ranges) of polynomials over finite fields have been studied by various authors, and many interesting results have been proved (see [3, pages 379–381]). Note that if $f(X)$ is a polynomial with integer coefficients, the formula $(\exists Y)(X = f(Y))$ will define in every finite field $F_q$ the value set of the function from $F_q$ to $F_q$ induced by $f$. The value sets of the discrete exponentials are no less interesting. For example, if $a > 1$ is an integer that is not a square, Artin’s conjecture for primitive roots [4] implies that the range of the function $t \rightarrow a^t \pmod{p}$ has $p - 1$ elements for infinitely many primes $p$. In the present note, we investigate the ranges of exponential functions

$$\exp_a : \mathbb{Z} \rightarrow F_p, \quad \exp_a(t) = a^t \pmod{p}, \quad (1.1)$$

from the point of view of definability. Note that the range of $\exp_a : \mathbb{Z} \rightarrow F_p$ coincides with $\langle a \rangle$, the cyclic subgroup of $F_p^*$ generated by $a$ (modulo $p$). Our main result will be the following.

**Theorem 1.1.** Let $a > 1$ be a fixed integer. Then there is no formula $\phi(X)$ in one free variable $X$, written in the first-order language of rings, such that for any prime $p$ with $\gcd(a, p) = 1$, the set of all elements in the finite prime field $F_p$ satisfying $\phi$ coincides with the range of the discrete exponential $\exp_a : \mathbb{Z} \rightarrow F_p$.

Here is a brief outline of the proof. We will first prove a result (Theorem 2.1) concerning the existence of primes with respect to which a fixed integer $a > 1$ has sufficiently small orders. This, in conjunction with a seminal result of Chatzidakis et al. [1] on definable subsets over finite fields, will lead to the proof of Theorem 1.1.
2. Small orders modulo $p$. In what follows, we will prove that there exist infinitely many primes with respect to which a given integer $a > 1$ has “small order.” More precisely, the following result holds true.

**Theorem 2.1.** Let $a > 1$ be an integer. Then, for every $\varepsilon > 0$, there exist infinitely many primes $q$ such that $\text{ord}_q(a)$, the order of $a$ modulo $q$, satisfies

$$\text{ord}_q(a) < q\varepsilon.$$  

**Proof.** Let $k$ be an integer satisfying

$$\frac{1}{k} < \varepsilon,$$  

and let $p$ be a prime satisfying

$$p > a,$$  

$$p \equiv 1 \pmod{(k+1)!}. $$

Due to Dirichlet’s theorem on primes in arithmetic progressions [2], there are infinitely many primes $p$ satisfying (2.3) and (2.4). We select a prime $q$ with the property

$$q \mid 1 + a + a^2 + \cdots + a^{p-1}. $$

Note that both $p$ and $q$ are necessarily odd. Since from (2.5) it follows that

$$a^p \equiv 1 \pmod{q},$$

the order $\text{ord}_q(a)$ can be either 1 or $p$. We will rule out the possibility $\text{ord}_q(a) = 1$. Indeed, if $\text{ord}_q(a) = 1$, then

$$q \mid a - 1.$$  

On the other hand, $1 + X + X^2 + \cdots + X^{p-1} = (X - 1)Q(X) + p$ with $Q(X)$ a polynomial with integer coefficients, and therefore

$$1 + a + a^2 + \cdots + a^{p-1} = (a - 1)Q(a) + p.$$  

From (2.5), (2.7), and (2.8) it follows $q \mid p$ and, since $p, q$ are primes, $q = p$. This, together with (2.7), leads us to $p \mid a - 1$, and therefore $a > p$, which contradicts assumption (2.3). This leaves us with

$$\text{ord}_q(a) = p.$$  

From (2.9) and from $a^{q-1} \equiv 1 \pmod{q}$ it follows that $p \mid q - 1$, so that

$$q = tp + 1.$$
for some positive integer \( t \). We will show that \( t > k \), so that

\[ q > kp + 1. \tag{2.11} \]

Indeed, we assume, for contradiction, that \( t \leq k \). From (2.4), we get \( p = (k+1)!s + 1 \) for some positive integer \( s \). Then

\[ q = tp + 1 = t((k+1)!s + 1) + 1 = t(k+1)!s + (t + 1). \tag{2.12} \]

Note that \( t + 1 \) is, under the assumption \( t \leq k \), a divisor of \( (k + 1)! \). Then, from (2.12), \( q \) will be a multiple of \( t + 1 \), a contradiction, since \( 2 \leq t + 1 < q \). Thus, (2.11) holds true and, consequently, since \( 1/k < \epsilon \), we get

\[ \frac{\text{ord}_q(a)}{q} = \frac{p}{q} < \frac{p}{kp + 1} < \frac{1}{k} < \epsilon, \tag{2.13} \]

which implies

\[ \liminf \frac{\text{ord}_q(a)}{q} = 0, \tag{2.14} \]

where the infimum is taken over all primes \( q > a \). This completes the proof of Theorem 2.1.

\[ \square \]

3. Proof of the main result. We now proceed to the proof of Theorem 1.1. We will use the following result which is a corollary of the main theorem in [1, page 108].

**Theorem 3.1.** If \( \phi(X) \) is a formula in the first-order language of rings, then there are constants \( A, C > 0 \), such that for every finite field \( K \), either \( |\phi(K)| \leq A \) or \( |\phi(K)| \geq C|K| \), where \( \phi(K) \) is the set of elements of \( K \) satisfying \( \phi \).

We are now ready to proceed to the proof of Theorem 1.1. Assume, for contradiction, that for some integer \( a > 1 \) there exists a first-order formula \( \phi(X) \) in the language of rings such that for every prime \( p \nmid a \), we have

\[ \phi(F_p) = \exp_a(F_p). \tag{3.1} \]

From (3.1) we get

\[ |\phi(F_p)| = \text{ord}_p(a) \tag{3.2} \]

for all \( p \nmid a \). Clearly,

\[ \text{ord}_p(a) > \log_a(p) \tag{3.3} \]

for all \( p \nmid a \). From (3.2), (3.3), and Theorem 3.1, it follows that for every large enough prime \( p \), we have

\[ \text{ord}_p(a) \geq Cp. \tag{3.4} \]
Clearly, (3.4) is in contradiction to Theorem 2.1 proved above, which implies that
\[ \liminf \frac{\text{ord}_p(a)}{p} = 0. \] (3.5)

**Remark 3.2.** From Theorem 1.1, it follows as an immediate corollary that, if \( a > 1 \) is a fixed integer, then there is no first-order formula \( \phi(X) \) in the first-order language of rings, such that for any prime \( p \), the set of all elements in \( F_p \) satisfying \( \phi \) is \( \{a^t \mod p \mid t \geq 1\} \). Indeed, assuming such a formula exists, it would define in any \( F_p \) with \( \gcd(a, p) = 1 \) the range of the discrete exponential \( \exp_a : \mathbb{Z} \to F_p \).

**Acknowledgment.** The authors wish to thank the anonymous referees for the helpful comments.

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