On a class of recurrent sequences based on the greatest prime factor function

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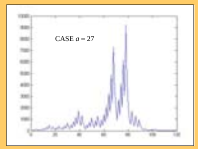
MOTIVATION: THE COLLATZ CONJECTURE

$$x_1 = a, \quad x_{N+1} = \begin{cases} 3x_N + 1, & \text{if } x_N \text{ is odd} \\ \frac{1}{2}x_N, & \text{if } x_N \text{ is even} \end{cases} \Rightarrow \{x_N\} \text{ eventually enters the cycle } 1,4,2,1,4,2,\ldots$$

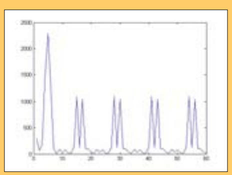


We use the greatest prime factor function to generate a class of sequences of primes which appear to exhibit a similar property of being ultimately periodic.

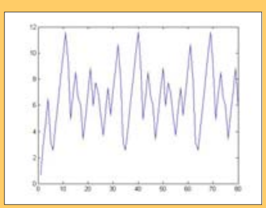
A "*linear* GPF *sequence*" will be a prime sequence of the following form:

$\operatorname{gpf}(n)$ = largest prime factor of n

$$\begin{cases} x_1 = p \text{ (some prime)} \\ x_{n+1} = \operatorname{gpf}(ax_n + b), n \geq 1 \\ (a, b \geq 1 \text{ – fixed integers}) \end{cases} \longleftarrow \operatorname{GPF}(a,b)$$



$a = 8, b = 1, x_1 = 293$
PERIOD:
11,89,31,83,19,17,137,1097,131,1049,109,97,37



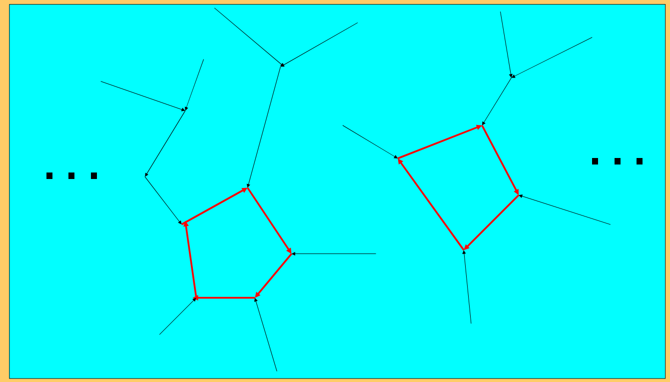
$a = 6, b = 5, x_1 = 2$ – logarithmic plot
PERIOD:
23, 13, 83, 503, 3023, 18143, 108863, 9749, 137, 827, 4967, 727, 397, 31, 191, 1151, 6911, 367, 2207, 1019, 211, 41, 251, 1511, 193, 1163, 6983, 41903, 2441

THE GPF CONJECTURE:
All linear GPF sequences are ultimately periodic

ASSOCIATED DIGRAPHS $D(a,b)$

Vertices – all primes.

For every prime p, an edge $p \to \operatorname{gpf}(ap+b)$.



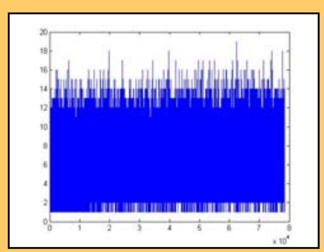
Graphical form of the GPF conjecture: the expected form of D(a,b)
No matter where we start from, the arrow flow leads to a limit cycle.

MULTIPLE LIMIT CYCLES ARE POSSIBLE!

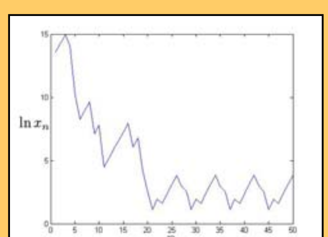
$a = 16, b = 1$

$x_1 = 2 \to$ PERIOD: 37,593,3163,229, 733, 317,89,19,61,977,193,3089,659

$x_1 = 47 \to$ PERIOD: 251, 103, 97, 1553



EXAMPLE: $D(2,1)$ [edges: $p \to \operatorname{gpf}(2p+1)$]
LIMIT CYCLE: 3,7,5,11,23,47,19,13
Plot shows the number of iterations needed to reach the limit cycle for primes $< 10^6$.
Among the primes $< 10^6$, 779111 requires the maximum number of iterations (19).



Logarithmic plot of the first 50 terms of the sequence $\{x_N\}_{N\geq 1} \in \operatorname{GPF}(2,1)$ with $x_1 = 779111$

RESULTS PROVED IN CONNECTION TO THE GPF CONJECTURE

THEOREM 1: The GPF conjecture is true for $a = 1$. That is, every sequence of primes satisfying a recurrence of the form $x_{n+1} = \operatorname{gpf}(x_n + b)$ is ultimately periodic.

THEOREM 2: If the GPF conjecture is true for a and b, and if $k \geq 1$, then it is also true for ka and kb.

In other words to prove the GPF conjecture it will be enough to show that is $\gcd(a,b)$=1 then every sequence of primes satisfying $x_{n+1} = \operatorname{gpf}(ax_n + b)$ is ultimately periodic.

Other open problems in connection to the GPF conjecture that we find interesting:

(A) Is the number of limit cycles in any digraph $D(a,b)$ finite?

(B) Is the in-degree of every vertex in $D(1,1)$ infinite? Note that in $D(1,1)$, $p \to 2$ is an edge if and only if p is a Mersenne prime, so (B) would be a generalization of the conjecture about the existence of infinitely many Mersenne primes!

REFERENCES
1. Caragiu, M. and Scheckelhoff, L., The Greatest Prime Factor and Related Sequences, JP J. of Algebra, Number Theory and Appl. 6 (2006) , 403 - 409.
2. Scheckelhoff, L., GPF Sequences, Senior Capstone Project, Ohio Northern University, 2006.

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All computational evidence for the GPF conjecture was obtained by using MATLAB.