Robinson's theorem in connection with a Putnam problem

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PROBLEM "A": Let $x_1, x_2, \ldots, x_{2n+1}$ be elements of a field K of characteristic zero such that if any one of them is removed, the remaining ones can be divided into two sets of n integers with equal sums. Prove that $x_1 = x_2 = \ldots = x_{2n+1}$.

SOLUTION TO PROBLEM "A"

STEP I. Proof of the case in which $x_1, x_2, \ldots, x_{2n+1}$ are non-negative integers (the original Putnam B1/1973 problem):

(1) The "$2n+1$ property" implies that $x_1, x_2, \ldots, x_{2n+1}$ have the same parity.

(2) Adding or multiplying $x_1, x_2, \ldots, x_{2n+1}$ by the same arbitrary constant does not change whether or not the $2n+1$ property holds.

(3) Use (1) and (2) to conclude the proof by performing induction on $\max(x_1, x_2, \ldots, x_{2n+1})$.

STEP II. Proof of the case in which $x_1, x_2, \ldots, x_{2n+1}$ are rational numbers: use (1) to perform a reduction to the case in which $x_1, x_2, \ldots, x_{2n+1}$ are nonnegative integers.

STEP III. Since K has characteristic zero, K is a vector space over the field of rationals. We consider a basis for this vector space and notice that the $2n+1$ property must hold over components. Applying STEP II, we conclude that fields of characteristic zero satisfy the $2n+1$ property.

Since every field of finite characteristic p is a vector space over its finite prime subfield F_p, arguing similarly to the earlier case, we can say that the $2n+1$ property *holds for every field of characteristic p only if it holds for* F_p.

PROBLEM "B": For a given integer $n \ge 1$ find all primes p such that the $2n+1$ property is valid in F_p (and thus, in every field of characteristic p).

To address problem "B" we will take a model-theoretic look at problem "A".

FT = The (finite) set of first-order axioms for the theory of fields.

FT_0 = The (infinite) set of first-order axioms for the theory of fields of characteristic zero. $FT_0 = FT \cup \{\underbrace{1+1+\ldots+1}_{n \text{ terms}} \ne 0,\ n \ge 1\}$

The "$2n+1$ property" is elementary, that is, can be expressed as a first-order statement Φ_n of the form $(x_1)\ldots(x_{2n+1})\big(D(x_1, x_2, \ldots, x_{2n+1}) \to E(x_1, x_2, \ldots, x_{2n+1})\big)$ where $E(x_1, x_2, \ldots, x_{2n+1})$ expresses the equality between $x_1, \ldots, x_{2n+1}$, while $D(x_1, x_2, \ldots, x_{2n+1})$ is a conjunction of $2n+1$ formulas $R_1, \ldots, R_{2n+1}$ with each R_i being a disjunction, over all n-element subsets S of $\{x_1, x_2, \ldots, x_{2n+1}\} - \{x_i\}$, of equalities between the sum of x_j with j in S and the sum of x_j with j in the complement of S with respect to $\{x_1, x_2, \ldots, x_{2n+1}\} - \{x_i\}$.

ROBINSON's THEOREM. If an elementary statement ψ in the first-order language of rings is true in every field of characteristic zero, then ψ is true in every field of large enough finite characteristic. This follows from the finitistic character of the logical proof and the structure of the axiom list FT_0. It also can be seen as an application of the compactness theorem.

CONSEQUENCE: Φ_n (the $2n+1$ property) is valid in every field of characteristic p if p is large enough.

DEFINITION: Let $P(n)$ be the smallest prime such that Φ_n is true in every field of characteristic $p \ge P(n)$. We will provide bounds for the function $P(n)$.

PROPOSITION 1. If $2n+1 \ge p \ge 5$ then Φ_n does not hold in F_p.
PROOF. Consider the list $\underbrace{0, 0, \ldots, 0}_{2n-p+2}, 1, 2, \ldots, p-1$ of elements of F_p.

PROPOSITION 2. If $p = 3$ and $n \ge 2$ then Φ_n does not hold in F_p.
PROOF. Consider the list $\underbrace{0, 0, \ldots, 0}_{2n-3}, 1, 1, 2, 2$ of elements of F_p.

PROPOSITION 3. Φ_n holds in the field K iff every $(2n+1)\times(2n+1)$ matrix A with zeros on the main diagonal and with n 1's and n -1's on each row has rank $2n$ over K.

THEOREM 4. $P(n) < 2(2n-1)^n$.

PROOF. Let A be a $(2n+1)\times(2n+1)$ matrix A with zeros on the main diagonal and with n 1's and n -1's on each row. According to [1], the principal minors of A are not zero. Indeed, the square of such a minor M is congruent mod 2 to the $2n \times 2n$ identity matrix. By Hadamard's inequality, $|\det(M)| \le (2n-1)^n$. Thus, if $p > (2n-1)^n$, then $\det(M) \not\equiv 0 \pmod{p}$. $P(n)$ is therefore less than, or equal to, the smallest prime greater than $(2n-1)^n$, which is, by Bertrand's theorem, less than $2(2n-1)^n$.

THEOREM 5. If $n \ge 2$ then $2n+3 \le P(n) < 2(2n-1)^n$.

REFERENCES

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